NONISOTHERMAL FLOW OF A RAREFIED GAS IN A CIRCULAR CYLINDRICAL CHANNEL

V. M. Zhdanov and V. A. Zaznoba

UDC 533.6.011

Using a method which averages the system of moment equations over the channel section, the authors have obtained expressions for Poiseuille flow and thermal creep flow in a capillary at intermediate Knudsen numbers ($\mathrm{Kn} \leqslant 0.25$).

The analysis of slow gas flow in channels at arbitrary Knudsen number (Kn) is based, as a rule, on numerical or variational methods of solving the linearized kinetic Boltzmann equation with model collision integral [1-6].

Although the general formulation of the problem using an exact statement of the linearized Boltzmann collision operator is well known (see, e.g., [2]), correct results have been obtained only in the $\mathrm{Kn} \ll 1$ region in computing coefficients of viscous and thermal slip [7, 8].

In [9, 10] the problem of flow of a gas in a planar channel at intermediate Knudsen number ($\mathrm{Kn} \leqslant 0.25$) was solved using an exact Boltzmann collision integral, accounting for the effect of second-order slip, which corresponds to using the Barnett term in the expansion of the distribution function. Here, however, the structure of the Barnett term in [9] was chosen on the basis of the simple analogy with its form for the special case of the BGK model. In the analysis of [10] using the 20-moment approximation to the distribution function one can write the Barnett term more correctly and thereby improve the dependence of macroscopic flow quantities in a channel on the Knudsen number.

In the present work the method used in [10] is applied to solve the problem of flow of gas in a cylindrical capillary.

We consider slow flow of a gas in a circular cylindrical channel of radius R under the influence of small relative gradients of pressure ($k = p_0^{-1}\, dp/dz$) and temperature ($\tau = T_0^{-1}\, dT/dz$). One can then seek a solution for the molecular distribution function in the form

$$f(\vec{v}, r, z) = f_0 \left[1 + kz + \tau z \left(\beta v^2 - \frac{5}{2} \right) + \Phi(r, \vec{v}) \right], \tag{1}$$

$$f_0 = n_0 (\beta/\pi)^{3/2} \exp(-\beta v^2), \quad \beta = m/2k_B T_0.$$

Here the subscript 0 corresponds to parameters of the absolute Maxwellian distribution, and $\Phi(r, \vec{v})$ is a nonequilibrium addition to the distribution function, determined from the linearized kinetic Boltzmann equation [11]

$$v_r \frac{\partial \Phi}{\partial r} + v_z k + v_z \tau \left(\beta v^2 - \frac{5}{2} \right) + \frac{v_\varphi^2}{r} \frac{\partial \Phi}{\partial v_r} - \frac{v_r v_\varphi}{r} \frac{\partial \Phi}{\partial v_\varphi} = L(\Phi). \tag{2}$$

To solve the problem, as in [10], we restrict ourselves to a set of moment equations which correspond to the 20-moment Grad approximation [12] in going to the continuum limit (the flow region close to the wall).

Multiplying Eq. (2) successively by $\Psi(\vec{c}) \exp(-c^2)$, where $\Psi = c_i,\ c_i c_j - \frac{1}{3} c^2 \delta_{ij},\ c_i \left(c^2 - \frac{5}{2} \right)$

Moscow Institute of Engineering Physics. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 44, No. 5, pp. 772-779, May, 1983. Original article submitted December 22, 1981.

0022-0841/83/4405-0524$07.50

and by $c_ic_jc_k - \frac{1}{5}c^2(c_i\delta_{jk} + c_j\delta_{ik} + c_k\delta_{ij})$, and $\vec{c} = \beta^{1/2}\vec{v}$, and integrating over velocities, we arrive at equations of moment type

$$\frac{1}{r}\frac{\partial}{\partial r} r p_{rz} + p_0 k = 0, \tag{3}$$

$$\beta^{1/2}\frac{\partial}{\partial r}\left(m s_{zrr} + \frac{2}{5}q_z + p_0 u_z\right) + \frac{m}{r}(s_{zrr} - s_{z\varphi\varphi}) = \frac{1}{l}p_{rz}, \tag{4}$$

$$\beta^{-1/2}\frac{1}{r}\frac{\partial}{\partial r} r\left(\Pi_{zrrr} + \Pi_{zr\varphi\varphi} + \Pi_{zrzz} - \frac{5}{2}p_{rz}\right) + \frac{5}{2}\tau p_0 = -\frac{4}{3l}q_z, \tag{5}$$

$$\beta^{-1/2}\frac{1}{r}\frac{\partial}{\partial r} r(4\Pi_{zrrr} - \Pi_{zr\varphi\varphi} - \Pi_{zrzz}) - \frac{10}{r}\Pi_{zr\varphi\varphi} = -\frac{15}{2l}m s_{zrr}, \tag{6}$$

$$\beta^{-1/2}\frac{1}{r}\frac{\partial}{\partial r} r(4\Pi_{zr\varphi\varphi} - \Pi_{zrrr} - \Pi_{zrzz}) + \frac{10}{r}\Pi_{zr\varphi\varphi} = -\frac{15}{2l}m s_{z\varphi\varphi}, \tag{7}$$

$$s_{zrr} + s_{z\varphi\varphi} + s_{zzz} = 0, \tag{8}$$

in which the values of the hydrodynamic velocity u_z, the viscous stress tensor p_{rz}, the heat flux q_z and the moments of third order s_{ijk} and fourth order Π_{ijkl} are determined by the expressions:

$$\left\|\begin{array}{c} u_i \\ p_{ij} \\ q_i \\ m s_{ijk} \\ \Pi_{ijkl} \end{array}\right\| = 2p_0\beta^{-1/2}\pi^{-3/2}\int \left\|\begin{array}{c} (2p_0)^{-1}c_i \\ \beta^{1/2}\left(c_ic_j - \frac{1}{3}c^2\delta_{ij}\right) \\ \frac{1}{2}c_i\left(c^2 - \frac{5}{2}\right) \\ c_ic_jc_k - \frac{1}{5}c^2(c_i\delta_{jk} + \\ + c_j\delta_{ik} + c_k\delta_{ij}) \\ \beta^{1/2}c_ic_jc_kc_l \end{array}\right\| \Phi\exp(-c^2)\,d\vec{c}\,. \tag{9}$$

Here $l = [\eta]_1/p_0\beta^{1/2}$ is the average mean free path, and the viscosity coefficient $[\eta]_1$ corresponds to the first approximation in the expansion in terms of Sonin polynomials in the Chapman–Enskog method [13].

Far from the walls the system of equations (3)-(8) must correspond to the usual 20-moment Grad approximation [12]. Following linearization, allowing for the smallness of the quantities u_z, p_{rz}, q_z, s_{ijk}, we can write the distribution function in this region in the form

$$f^a(\vec{c}, r, z) = f_0\left[1 + kz + \tau z\left(c^2 - \frac{5}{2}\right) + \Phi^a(\vec{c}, r)\right], \tag{10}$$

$$\Phi^a(\vec{c}, r) = 2\beta^{1/2}c_z u_z^a + 2c_zc_r p_0^{-1}p_{rz}^a + \frac{4}{5}\beta^{1/2}p_0^{-1}q_z^a c_z\left(c^2 - \frac{5}{2}\right) + 2\beta^{1/2}p_0^{-1}mc_z\left(s_{zrr}^a c_r^2 + s_{z\varphi\varphi}^a c_\varphi^2 + \frac{1}{3}s_{zzz}^a c_z^2\right).$$

Substituting Eq. (10) into the definition of Π_{ijkl} from Eq. (9) and integrating over the velocities, we have

$$\Pi_{zrrr}^a = \frac{3}{2}p_{rz}^a;\quad \Pi_{zr\varphi\varphi}^a = \frac{1}{2}p_{rz}^a;\quad \Pi_{zrzz}^a = \frac{3}{2}p_{rz}^a\,. \tag{11}$$

The solution of the system of equations (3)-(8), allowing for Eq. (11), yields explicit expressions for u_z^a, p_{rz}^a, q_z^a, s_{ijk}^a and by substituting these into Eq. (10) one can obtain an expression for the asymptotic correction to the distribution function in the form

$$\Phi^a(\vec{c}, r) = 2c_z\left[\beta^{1/2}u_z^a(R) - \frac{k}{4l}(R^2 - r^2)\right] - krc_rc_z -$$

$$-\frac{3}{2}\tau lc_z\left(c^2 - \frac{5}{2}\right) + \frac{3}{5}lkc_z\left\{\left(c^2 - \frac{5}{2}\right) + \frac{4}{9}\left[\frac{5}{2}(c_r^2 + c_\varphi^2) - c^2\right]\right\}. \tag{12}$$

The last term of this expression is the Barnett correction to the distribution function for cylindrical geometry problem considered here.

The corresponding solution for the planar problem leads to a Barnett correction of the type

$$\frac{2}{5}\,\mathrm{Pr}^{-1}lkc_z\left[\left(c^2 - \frac{5}{2}\right) + \frac{4}{9}(5c_x^2 - c^2)\right]; \quad \mathrm{Pr} = \frac{2}{3}. \tag{13}$$

We note that Eq. (13) differs considerably from the form of the analogous correction obtained in [1] by the method of elementary solutions for the BGK collision integral model. Therefore, the choice in [9] of a Barnett term for an arbitrary molecular interaction law in the form $\mathrm{Pr}^{-1}(c_x^2 - 1)$, where $\mathrm{Pr} = 2/3$ instead of $\mathrm{Pr} = 1$ [1], is incorrect.

We turn now to the general equations (3)-(8). We note that it follows from the solution of Eq. (3) that the relation

$$p_{rz}(r) = -\frac{krp_0}{2} \tag{14}$$

is valid in the entire flow region. Substituting this value into Eq. (4), integrating the relation obtained with respect to r, and averaging over the channel section, we obtain

$$ms_{zrr}(R) - \frac{m}{2}(\langle s_{zrr}\rangle + \langle s_{z\varphi\varphi}\rangle) + \frac{2}{5}[q_z(R) - \langle q_z\rangle] + p_0[u_z(R) - \langle u_z\rangle] = \beta^{-1/2}\frac{kR^2p_0}{8l}, \tag{15}$$

where $\langle Q\rangle = \frac{2}{R^2}\int\limits_0^R Q(r)\,rdr$. Also averaging Eqs. (5)-(7) over the channel section, we have

$$\frac{2}{R}[\Pi_{zrrr}(R) + \Pi_{zr\varphi\varphi}(R) + \Pi_{zrzz}(R)] + \frac{5}{2}kp_0 + \frac{5}{2}\tau p_0 = -\frac{4}{3l}\beta^{1/2}\langle q_z\rangle, \tag{16}$$

$$\frac{2}{R}[3\Pi_{zrrr}(R) + 3\Pi_{zr\varphi\varphi}(R) - 2\Pi_{zrzz}(R)] = -\frac{15}{2l}m\beta^{1/2}(\langle s_{zrr}\rangle + \langle s_{z\varphi\varphi}\rangle). \tag{17}$$

To determine the unknown quantities at the channel wall we use the approximate method of Loyalka [7]. We introduce the incident and reflected molecule distribution function such that $\Phi = \Phi^+$ for $c_r > 0$ and $\Phi = \Phi^-$ for $c_r < 0$. According to Eq. (12), for the functions $\Phi^\pm$ at $r = R$, accounting for the usual Maxwellian condition of reflection of molecules at the wall, we have

$$\Phi^+(\vec{c}, R) = 2\beta^{1/2}c_za - kRc_rc_z - \frac{3}{2}\tau lc_z\left(c^2 - \frac{5}{2}\right) +$$

$$+\frac{3}{5}lkc_z\left\{\left(c^2 - \frac{5}{2}\right) + \frac{4}{9}\left[\frac{5}{2}(c_r^2 + c_\varphi^2) - c^2\right]\right\}, \quad c_r > 0, \tag{18}$$

$$\Phi^-(\vec{c}, R) = (1 - \varkappa)\Phi^+(-c_r, c_\varphi, c_z, R), \quad c_r < 0,$$

where $\varkappa$ is the fraction of molecules undergoing diffuse reflection at the wall, and instead of $u_z^a(R)$ we introduce the arbitrary constant a, defined from the condition that the tangential component of the momentum flux of gas at the wall be conserved.

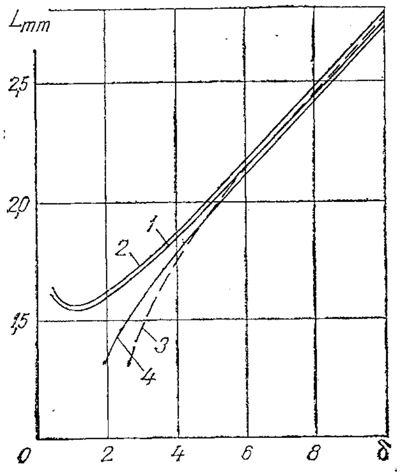

Fig. 1

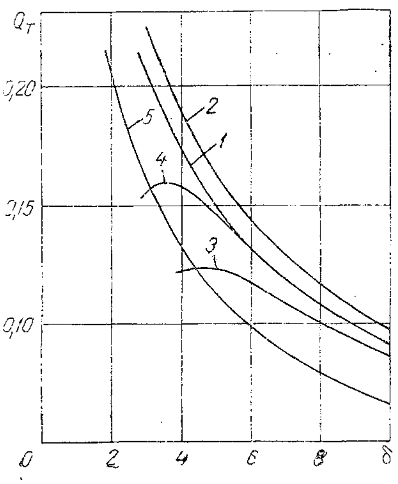

Fig. 2

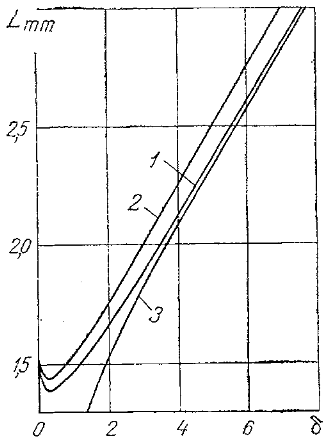

Fig. 3

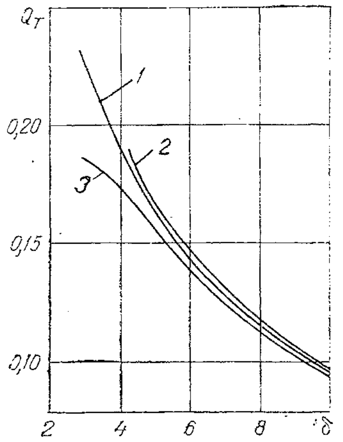

Fig. 4

Fig. 1. Poiseuille flow as a function of inverse Knudsen number for a planar channel: 1) BGK [1]; 2) S model [6]; 3) [9]; 4) [10].

Fig. 2. Thermal creep flow as a function of inverse Knudsen number for a planar channel: 1) modified BGK [3]; 2) S model [6]; 3) [9]; 4) [10]; 5) BGK [2].

Fig. 3. Poiseuille flow as a function of inverse Knudsen number for a circular cylindrical channel: 1) BGK [1], S model [5]; 2) the Knudsen formula; 3) the present work.

Fig. 4. Thermal creep flow as a function of inverse Knudsen number for a circular cylindrical channel: 1) modified BGK [3]; 2) S model [5]; 3) the present work.

Using condition (14) and the definition of p_{rz} by Eq. (9) at the channel wall, and after computing the appropriate integrals and using Eq. (18), we obtain

$$a=\beta^{-1/2}\left[-\frac{(2-\varkappa)}{\varkappa}\frac{\sqrt{\pi}}{4}kR-\frac{kR}{4\delta}+\frac{3\tau R}{8\delta}\right], \tag{19}$$

where $\delta = R/l$ is the Knudsen number.

Using Eq. (19) to calculate the quantities at the channel wall appearing in Eqs. (15)-(17) and of interest to us, we find the values, averaged over the channel section, of the gas velocity and the heat flux

$$\langle u_z\rangle=-\beta^{-1/2}kR\left[\frac{\delta}{8}+\frac{(2-\varkappa)}{2}\left(\frac{1}{\sqrt{\pi}}+\frac{2-\varkappa}{\varkappa}\frac{\sqrt{\pi}}{4}\right)+\frac{\varkappa}{4\delta}-\frac{5\varkappa}{6\sqrt{\pi}\,\delta^2}\right]-\beta^{-1/2}\tau R\left[-\frac{3}{16}(2+\varkappa)\frac{1}{\delta}+\frac{3\varkappa}{2\sqrt{\pi}\,\delta^2}\right], \tag{20}$$

$$\langle q_z \rangle = 2p_0\beta^{-1/2}\left\{-kR\left[-\frac{3}{32}(2+\varkappa)\frac{1}{\delta}+\frac{3\varkappa}{4\sqrt{\pi}\,\delta^2}\right]-\tau R\left[\frac{15}{16\delta}-\frac{27\varkappa}{16\sqrt{\pi}\,\delta^2}\right]\right\}.$$

We now introduce the dimensionless quantities $J_m^* = J_m/mJ_0 = 2\beta^{1/2}\langle u_z \rangle$, $J_q^* = J_q/k_B T_0 J_0 = 2\beta^{1/2} p_0^{-1}\langle q_z \rangle$, where J_m and J_q are the corresponding averaged fluxes of mass and heat, reduced to unit area of channel section, and $J_0 = n_0/2\beta^{1/2}$. Then, in accordance with the principles of the thermodynamics of irreversible processes [14], $J_m^* = -L_{mm}kR - L_{mq}\tau R$, $J_q^* = -L_{qm}kR - L_{qq}\tau R$. General expressions for the coefficients L_{ik} follow from Eq. (20). We note that the cross coefficients L_{ik} satisfy the Onsager reciprocity relation ($L_{mq} = L_{qm}$). In particular, for the case of complete diffuse reflection of molecules ($\varkappa = 1$), these coefficients take the form

$$L_{mm} = \frac{\delta}{4} + \sigma + \frac{1}{2\delta} - \frac{5}{3\sqrt{\pi}\,\delta^2}, \quad L_{mq} = L_{qm} = -\frac{a_T}{\delta} + \frac{3}{\sqrt{\pi}\,\delta^2}, \quad L_{qq} = \frac{15}{4\delta} - \frac{27}{4\sqrt{\pi}\cdot\delta^2}, \tag{21}$$

where $\sigma = 1.0073$ and $a_T = 1.125$ are the coefficients of viscous and thermal slip, whose values coincide with the results obtained in [8] using the variational method.

For comparison we present the corresponding expressions for L_{ik} in the case of a planar channel [10] ($\delta = d/l$, where d is the channel width):

$$L_{mm} = \frac{\delta}{6} + \sigma + \frac{5}{6\delta} - \frac{3}{\sqrt{\pi}\,\delta^2}, \quad L_{mq} = L_{qm} = -\frac{a_T}{\delta} + \frac{7}{2\sqrt{\pi}\,\delta^2}, \quad L_{qq} = \frac{15}{4\delta} - \frac{27}{4\sqrt{\pi}\,\delta^2}. \tag{22}$$

We note that Eqs. (21) and (22) are valid for an arbitrary molecular interaction law. Analogous results can be obtained also for the BGK collision integral model. A comparison of these with results of an exact numerical solution for this model shows that the present method gives satisfactory agreement with the exact method in the region $\mathrm{Kn} \leqslant 0.25$.

Figures 1-4 show the quantities L_{mm} (describing isothermal Poiseuille flow of gas in a channel) and $Q_T = -L_{mq}$ (describing thermal creep flow) for a planar and a cylindrical channel, as a function of the inverse Knudsen number δ. Here also we show for comparison the results of calculations using different collision integral models. As can be seen from the curves presented, the results of computing the coefficient L_{mm} with the BGK model and the S model for $\mathrm{Kn} \leqslant 0.25$ agree satisfactorily with our results that do not use the model representations. We note that the curves corresponding to the Knudsen interpolation formula describing the experimental data for Poiseuille flow in a cylindrical channel fall above the computed curves by an average of 5%, which is evidently connected with the difference in the momentum accommodation coefficient at the wall from unity in these experiments.

In regard to the thermal creep flow, the results of our calculations of Q_T in the region $\mathrm{Kn} \leqslant 0.2$ agree well with the calculations for the modified BGK model (using two rarefaction parameters defined for the gas viscosity and thermal conductivity). Here the values for the S model are somewhat higher, and the results of [9] for a planar channel are noticeably lowered due to incorrect choice of the Barnett term in the distribution function.

NOTATION

Kn, Knudsen number; Pr, Prandtl number; k, τ, relative gradients of pressure and temperature; r, φ, z, variables in a cylindrical coordinate system; u_z, hydrodynamic velocity; p_{rz}, viscous stress tensor; q_z, heat flux; s_{ijk}, Π_{ijkl}, moments of third and fourth order; l, mean free path; R, channel radius; d, channel width; δ, inverse Knudsen number; J_m^*, J_q^*, averaged dimensionless fluxes of mass and heat, referenced to unit area of channel cross section; $\varkappa$, fraction of molecules undergoing diffuse reflection at the wall.

LITERATURE CITED

1. K. Cherchin'yanin, Theory and Applications of the Boltzmann Equation [in Russian], Mir, Moscow (1978).

2. S. K. Loyalka, "Kinetic theory of thermal transpiration and the mechanocaloric effect, J. Chem. Phys., 55, No. 9, 4497 (1971).
3. S. K. Loyalka, "Kinetic theory of thermal transpiration and the mechanocaloric effect, 2," J. Chem. Phys., 63, No. 9, 4054 (1975).
4. B. T. Porodnov and F. T. Tukhvetov, "Theoretical investigation of nonisothermal flow of a rarefied gas in a cylindrical capillary," Inzh.-Fiz. Zh., 36, No. 1, 86 (1979).
5. V. G. Chernyak, V. V. Kalinin, and P. E. Suetin, "Theory of thermomolecular pressure and the mechanocaloric effect in a cylindrical channel," Inzh.-Fiz. Zh., 37, No. 1, 91 (1979).
6. V. G. Chernyak, V. V. Kalinin, and P. E. Suetin, "Theory of the nonisothermal motion of a gas in a planar channel," Inzh.-Fiz. Zh., 36, No. 6, 1059 (1979).
7. S. K. Loyalka, "Approximate method in the kinetic theory," Phys. Fluids, 14, No. 11, 2291 (1971).
8. S. K. Loyalka, "The slip problem for a simple gas," Z. Naturforsch., 26a, 964 (1971).
9. H. Lang, "Second-order slip effects in Poiseuille flow," Phys. Fluids, 19, No. 3, 366 (1976).
10. V. M. Zhdanov and V. A. Zabnova, "Nonisothermal flow of a gas mixture in a channel at intermediate Knudsen number," Zh. Prikl. Mat. Mekh., 45, No. 6, 1063 (1981).
11. M. N. Kogan, Dynamics of a Rarefied Gas, Plenum Publ.
12. H. Grad, "On the kinetic theory of rarefied gases," Commun. Pure Appl. Math., 2, No. 4, 331 (1949) [Mekhanika, 4 (14), 71 (1952); 5(15), 61 (1952)].
13. S. Chapman and T. G. Cowling, Mathematical Theory of Non-Uniform Gases, Cambridge Univ. Press (1970).
14. S. R. De Groot and P. Mazur, Non-equilibrium Thermodynamics, Elsevier (1962).

A METHOD OF SIMULTANEOUS MEASUREMENT OF THE SORET AND DIFFUSION COEFFICIENTS OF LIQUID SOLUTIONS

V. M. Buzmakov, A. Yu. Pinyagin, and A. F. Pshenichnikov

UDC 532.08;533.1/5.08

A method is developed for the simultaneous measurement of the Soret and diffusion coefficients under the conditions of a quasiisothermal process of damping of concentration disturbances in the measurement cell. Measurement results are presented for aqueous solutions of potassium chloride and sodium sulfate.

The principal methods of measuring the Soret coefficient, which characterizes the amount of thermodiffusional separation, presume the use of thermodiffusion columns or so-called convectionless cells. Optical methods are usually used to measure the concentration gradient in the cells. The latter do not introduce disturbances, permit one to reduce the size of the cell, and shorten the time of one test to several hours [1].

Finding the Soret coefficient from the conditions of a steady-state process is hindered by the fact that the optical signal is determined by the temperature and concentration fields simultaneously. In this case the contribution of the temperature field considerably exceeds the contribution of the concentration field. Therefore, measurement of the concentration difference is performed with considerable errors. The absence of precise information about the temperature dependence of the index of refraction of the mixture results in additional errors. Nonsteady methods, based on an analysis of the process of establishment of the concentration field under the action of an applied temperature difference, have the same drawbacks.

The accuracy in measuring the concentration gradients can be increased if the experiments are conducted under isothermal conditions. In this case the initial concentration difference is also created through thermodiffusion, but after the stationary regime is established the temperature at the boundaries of the measurement cell is made the same. Thanks to this, the time of relaxation of temperature disturbances is two orders of magnitude less

---

A. M. Gor'kii State University, Perm. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 44, No. 5, pp. 779-783, May, 1983. Original article submitted December 7, 1981.

0022-0841/83/4405-0529$07.50